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Gap solitons in Bose–Einstein condensates in linear and nonlinear optical lattices

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Abstract

Properties of localized states on array of BEC confined to a potential, representing superposition of linear and nonlinear optical lattices are investigated. For a shallow lattice case the coupled mode system has been derived. We revealed new types of gap solitons and studied their stability. For the first time a moving soliton solution has been found. Analytical predictions are confirmed by numerical simulations of the Gross–Pitaevskii equation with jointly acting linear and nonlinear periodic potentials.

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1. Introduction

The Bose–Einstein condensate in a linear periodic potential attracts a great attention for last years. Many fascinating phenomena like Josephson oscillations, macroscopic quantum tunnelling and localization, gap solitons, etc., have been predicted and observed in the experiments [1,2]. The description of these phenomena is based on the Gross–Pitaevskii (GP) equation governing the condensate wave function and having terms corresponding to the external linear (varying in space and time) potential and a mean field nonlinearity (taking into account many-body effects). The strength of the mean field nonlinearity is proportional to the atomic scattering length. We can imagine the BEC system, when the strength of two-body interaction is varying in space [3–6]. For a periodic variation of scattering length it leads to appearance of a nonlinear periodic potential. The ground state and dynamics of localized states of BEC under action of the nonlinear periodic potential have been studied recently in papers [7–9].

Periodic potential in BEC [10,11] can be generated by optical methods. Two types of optical lattices are considered: a linear periodic potential induced by the standing laser field [1] and a nonlinear optical lattice produced by two counter propagating laser beams with parameters near the optically induced Feshbach resonance (FR) [7,8]. Periodic variation of the laser field intensity in space by proper choice of the resonance detuning can lead to the spatial dependence of the scattering length $a_s(x)$ [11]. Then the nonlinear term in the GP equation becomes periodically modulated in space, leading to the generation of nonlinear optical lattice.

In real physical situations exact resonance condition is not attained and the potential represents the mixture of linear and nonlinear optical lattices. This type of periodic potential also can be produced by two pairs of counter propagating laser beams, when one pair produces linear optical lattice, and second pair—a nonlinear one. Another interesting system where the superposition of linear and nonlinear periodic potentials can be realized is the array of fermion–boson mixtures [12]. Gap solitons (GS) in a photorefractive crystal, considered recently in [13], also originate from the jointly acting linear and nonlinear periodic potentials. But the last system is different from ones considered here, since its nonlinearity has saturable nature.

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Two types of optical lattices can be distinguished: a shallow and a deep. The first type is realized for $V < E_R$, and the second one for $V > 5E_R$, where V is the strength of the periodic potential and $E_R = \hbar^2 k^2/2m$ is the recoil energy, k is the laser beam wavenumber.

The case of shallow linear optical lattice has been investigated in [14–16], using in particular the coupled-mode theory. This theory describes general properties of gap solitons, taking into account interaction of counter propagating waves. Analytical solutions have been found for the case of the Kerr nonlinearity in [14]. The modulational instability and generation of gap solitons have been predicted to emerge with both plane waves and wave packets travelling on optical lattice in [15,17,18].

In this work we will derive generalized coupled-mode system of equations, describing BEC under joint action of linear and nonlinear optical lattices. This system of coupled-mode theory has more complicated character and is analogous to the studied in the nonlinear optics for a deep Bragg grating case [19] and a nonlinear layered structures [20]. In the case of vanishing nonlinear lattice a periodic potential transforms into the standard linear one [21–23]. The gap soliton solutions of this modified coupled mode system are obtained and their stability is analyzed.

The emphasis is also given to the travelling gap soliton solutions. We obtain solutions for moving GS and study collisions between solitons having an inelastic character. In optics they have been experimentally observed in fibers with Bragg grating [24]. It will be interesting to observe the travelling GS in BEC with linear and nonlinear optical lattices.

The Letter is organized as follows. In Section 2 we describe the physical model for BEC under joint action of linear and nonlinear optical lattices, based on the optical manipulation of the scattering length by the optically induced Feshbach resonances. The system is described by the 1D Gross–Pitaevskii equation with linear and nonlinear periodic potentials. Starting from this GP equation, we derive the system of equations of a coupled-mode theory. The problem of existence of gap soliton solutions, when the atomic scattering length a_s is periodically modulated in space, and their stability is discussed in Sections 3, 4. We obtain analytical solutions for the gap solitons in this type of periodic potentials and confirm them by the direct numerical simulations of the full GP equation. The moving gap soliton solutions, their stability and the solitons collision are studied in Section 5. Main results of the Letter are summarized in Section 6.

2. The model. Coupled-mode equations

We consider here the dynamics of BEC under jointly acting linear and nonlinear optical lattices. First type of potential is induced by a periodic interference pattern from the counter propagating laser beams [1]. Second type of potential is produced by the optically induced Feshbach resonance [11]. According to this approach, the scattering length a_s can be optically manipulated, if the incident light is close to the resonance with one of the bound p levels of electronically excited molecules. Virtual radiative transitions of a pair of interacting atoms to this level can change value and/or reverse sign of the scattering length a_s. The theory is developed for ^{7}Li and ^{87}Rb cases.

The periodic variation of laser field intensity in the standing wave $I(x) = I_0 \cos^2(kx)$ produces periodic variation of the atomic scattering length a_s, i.e.

$$a_s(X) = a_{s0} + \alpha \frac{I}{\delta + \alpha I}, \tag{1}$$

where δ is the detuning. At large detuning from FR $a_s = a_{s0} + a_{s1}\cos^2(kx)$.

In the recent experiment with the ^{87}Rb condensate the scattering length has been optically manipulated [10]. The number of atoms was 10^6 and laser power was approximately $500\ \mathrm{W/cm^2}$. By choice of the laser power and detuning of the laser beam around the photo-association resonance the scattering length was changed over one order of magnitude, from $10a_0$ to $190a_0$ (a_0 is the Bohr radius).

The quasi 1D GP equation for the BEC wave function under action of superposition of a linear and nonlinear optical lattice is [7,8]:

$$\begin{aligned} i\hbar\psi_T = {} & -\frac{\hbar^2}{2m}\psi_{XX} + V_0\cos^2(kx)\psi \\ & + \left(g_{1D}^{(0)} + g_{1D}^{(1)}\cos^2(kx)\right)|\psi|^2\psi, \end{aligned} \tag{2}$$

where $g_{1D}^{(0,1)} = 2\hbar a_{s0,1}\omega_\perp$, and a_s is the atomic scattering length. Introducing dimensionless variables

$$\begin{aligned} &\epsilon = \frac{V_0}{2E_R}, \quad E_R = \frac{\hbar^2 k^2}{2m}, \qquad x = Xk, \\ &t = \frac{T}{T_0}, \quad T_0 = \frac{\hbar}{E_R}, \\ &u = \sqrt{\frac{2\hbar|\bar{a}_{s0}|\omega_\perp}{E_R}}\psi e^{-i\epsilon t}, \qquad \kappa = -\frac{a_{s1}}{2\bar{a}_{s0}}, \\ &\gamma_0 = -\frac{\bar{a}_{s0}}{|\bar{a}_{s0}|}, \quad \bar{a}_{s0} = a_{s0} + \frac{a_{s1}}{2}, \end{aligned}$$

we can rewrite the equation in the form

$$iu_t + u_{xx} + \left(\gamma_0 + \kappa\cos(2x)\right)|u|^2u - \epsilon\cos(2x)u = 0. \tag{3}$$

$\gamma_0 > 0$ corresponds to the attractive and $\gamma_0 < 0$—to the repulsive condensates, respectively. The parameter κ change sign, if we shift the phase of the nonlinear optical lattice relative to linear one by π. $\kappa > 0$ corresponds to the out-of-phase and $\kappa < 0$—to the in-phase nonlinear optical lattices, respectively. This form of the GP equation with optical lattice and periodic in space nonlinearity appears also at the description of an array of the Fermi–Bose mixture. The fermionic component modifies the effective optical lattice and introduce the nonlinear periodic potential for bosons, leading to this form of the GP equation for the wave function of bosons [12].

We will consider the shallow lattice case, when $\epsilon \ll 1$. In this case one can derive the system of coupled mode equations. Let us represent the field $u(x,t)$ in the form of the superposition of the forward- and backward-propagating waves

$$u(x,t) = \left(A(x,t)e^{ix} + B(x,t)e^{-ix}\right)e^{-it}. \tag{4}$$

Collecting terms at $e^{\pm ix}$ we obtain the system of coupled-mode equations

$$iA_t + 2iA_x + \gamma_0\big(|A|^2A + 2|B|^2A\big) - \frac{\epsilon}{2}B + \frac{\kappa}{2}\big(|B|^2B + 2|A|^2B + A^2B^*\big) = 0, \tag{5}$$

$$iB_t - 2iB_x + \gamma_0\big(|B|^2B + 2|A|^2B\big) - \frac{\epsilon}{2}A + \frac{\kappa}{2}\big(|A|^2A + 2|B|^2A + B^2A^*\big) = 0. \tag{6}$$

In comparison with the standard coupled-mode equations, this system includes terms corresponding to the periodic modulations of mean-field nonlinearity $\sim\kappa$.

Eqs. (5), (6) read as

$$iA(B)_t = \frac{\delta H}{\delta A^*(B^*)}.$$

The Hamiltonian H has the following form

$$H = \int dx\left[\frac{i}{2}\big(A^*A_x - AA_x^* - B^*B_x + BB_x^*\big) - \frac{\epsilon}{2}(BA^* + AB^*) + \frac{\gamma_0}{2}\big(4|A|^2|B|^2 + |A|^4 + |B|^4\big) + \frac{\kappa}{2}\big(|A|^2 + |B|^2\big)(BA^* + B^*A)\right]. \tag{7}$$

The number of atoms is conserved

$$N = \int_{-\infty}^{\infty} dx\,\big[|A|^2 + |B|^2\big]. \tag{8}$$

The cw solutions of the coupled-mode system are sought using the parametrization suggested in [25]

$$A = \frac{\alpha}{\sqrt{1+f^2}}e^{i(Qx-\Omega t)}, \qquad B = \frac{\alpha f}{\sqrt{1+f^2}}e^{i(Qx-\Omega t)}, \tag{9}$$

where $\alpha^2 = |A|^2 + |B|^2$ is the condensate density. Then we have

$$\Omega = -\frac{3\gamma_0}{2}\alpha^2 + \frac{\epsilon}{4}\frac{1+f^2}{f} - \frac{\kappa\alpha^2}{4}\frac{1+6f^2+f^4}{f(1+f^2)}, \tag{10}$$

$$Q = -\frac{\gamma_0\alpha^2}{4}\frac{1-f^2}{1+f^2} + \frac{\epsilon}{8}\frac{1-f^2}{f} - \frac{\kappa\alpha^2}{8}\frac{1-f^2}{f}. \tag{11}$$

When $\kappa = 0$ the result for a standard coupled-mode system is reproduced. When the nonlinearity is negligible we have the linear dispersion law $\Omega = \pm 2\sqrt{Q^2 + \epsilon^2/4}$. In dimension units

$$\Omega = \pm\omega_R\sqrt{\frac{Q^2}{k^2} + \frac{V_0^2}{32E_R^2}}, \quad \omega_R = \frac{E_R}{\hbar}.$$

The spectrum has the forbidden gap. The velocity of linear waves is $v = d\Omega/dQ = v_R(1-f^2)/(1+f^2)$, $v_R = \hbar k/m$. The parameter f describes the division of the total number of atoms between the backward- and forward-propagating matter waves. One can see that the velocity V inside the optical lattice tends to zero near the edges of the forbidden gap, when $f \to \pm 1$. The case $f > 0$ corresponds to the top of a band gap of the linear dispersion curve, and $f < 0$ to the bottom of a band gap.

Eqs. (10), (11) can be used to analyze the influence of nonlinearity on the dispersion properties of BEC in nonlinear optical lattice. As known the nonlinearity leads to the appearance of a loop on the upper branch of the dispersive curve, with $f > 0$. The critical value of the condensate density α can be found using the condition of $Q = 0$ with $|f| \neq 1$ [26]. Then we obtain

$$f_{\text{crit}} = \frac{\gamma_0\alpha^2}{(\epsilon - \kappa\alpha^2)} + \sqrt{\frac{\gamma_0^2\alpha^4}{(\kappa\alpha^2 - \epsilon)^2} - 1}.$$

We have $\alpha^2 > \epsilon/(\gamma_0 - \kappa)$. Thus periodic modulation in space of the scattering length leads to decreasing of the value of the loop in the dispersion curve. If the condensate is repulsive the loop appears on the lower branch ($f < 0$) and the loop size is also reduced by the nonlinear optical lattice.

3. Gap soliton in linear–nonlinear lattice

To find gap soliton solution let us look for the particular solution of the form $A, B \to (a(x), b(x))\exp(-i\omega t)$.

The symmetry of the system (5) allows to consider the case $a(x) = b^*(x)$ [27]. Then we get the equation for $a(x)$,

$$\omega a + 2ia_x + 3\gamma_0|a|^2a - \frac{\epsilon}{2}a^* + \frac{3}{2}\kappa|a|^2a^* + \frac{\kappa}{2}a^3 = 0. \tag{12}$$

The solution will be sought in the form

$$a(x) = \sqrt{U(x)}e^{-i\theta(x)/2}.$$

Substituting this expression into Eq. (12) we get the following system of equations

$$\theta_x = -\omega - 3\gamma_0 U + \frac{\epsilon}{2}\cos\theta - 2\kappa U\cos\theta, \tag{13}$$

$$U_x = \frac{\epsilon}{2}U\sin\theta - \kappa U^2\sin\theta. \tag{14}$$

The system has the Hamiltonian H_g and can be written in the form

$$\theta_x = \frac{\partial H_g}{\partial U}, \qquad U_x = -\frac{\partial H_g}{\partial\theta}, \tag{15}$$

where

$$H_g = -\omega U - \frac{3}{2}\gamma_0 U^2 + \frac{\epsilon}{2}U\cos\theta - \kappa U^2\cos\theta. \tag{16}$$

We will consider solutions that rapidly decay in the space, i.e. $U \to 0$, $|x| \to \infty$. In this case $H_g = 0$, that corresponds to separatrix in the phase-plane portrait (see Fig. 1).

Then for θ we have the following equation

$$\theta_x = \omega - \frac{\epsilon}{2}\cos\theta, \tag{17}$$

with U determined as

$$U = -\frac{2\omega - \epsilon\cos\theta}{3\gamma_0 + 2\kappa\cos\theta}. \tag{18}$$

Integrating Eq. (17) provided that $|\tan(\theta/2)| < \gamma$ we find

$$\cos\theta = \frac{1 - \gamma^2\tanh^2(\beta x)}{1 + \gamma^2\tanh^2(\beta x)}, \tag{19}$$

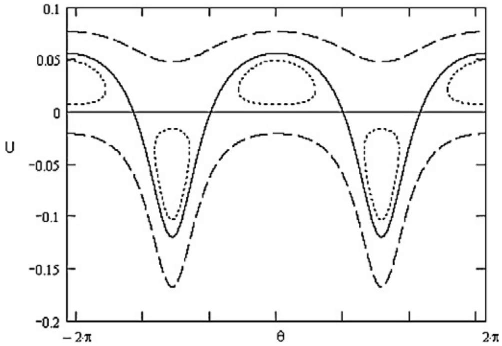

Fig. 1. Phase-plane portrait $U = U(\theta, H_g)$ at $\gamma_0 = 1$, $\epsilon = 0.2$, $\kappa = 1$ and $\omega = -0.04$. Solid line is for $H = 0$, dashed line for $H = -0.04$, dotted line for $H = 0.00085$.

where

$$\gamma = \sqrt{\frac{\epsilon - 2\omega}{\epsilon + 2\omega}}, \qquad \beta = \frac{\sqrt{\epsilon^2 - 4\omega^2}}{4}.$$

From Eq. (18) we obtain expression for the amplitude U,

$$U(x) = \frac{\epsilon - 2\omega}{(3\gamma_0 + 2\kappa)\cosh^2(\beta x) + (3\gamma_0 - 2\kappa)\gamma^2 \sinh^2(\beta x)}. \tag{20}$$

Finally the gap solution takes the form

$$u(x,t) = 2\sqrt{U}\cos\left(x - \frac{\theta(x)}{2}\right)e^{-i(\omega+1)t}, \tag{21}$$

with

$$-\epsilon/2 < \omega < \epsilon/2. \tag{22}$$

The gap soliton width is defined by the parameter ω and is inversely proportional to the optical lattice amplitude V_0. The amplitude of soliton is inversely proportional to the amplitude of modulations of the scattering length.

The number of atoms in the soliton is

$$N = \int_{-\infty}^{\infty} \left|u(x,t)\right|^2 dx = 2\int_{-\infty}^{\infty} U\, dx. \tag{23}$$

Let $\epsilon - 2\omega \ll \epsilon$. Then

$$U \approx \frac{\epsilon - 2\omega}{(3\gamma_0 + 2\kappa)\cosh^2(\beta x)},$$

i.e. the solution near the edge of the gap is the NLSE soliton. We can see that if $\kappa \neq 0$, the solution takes again the NLS form with the effective nonlinearity $\gamma_0 + \kappa/3$. Interesting case we have when the lattices are shifted by π, so $\kappa \to -\kappa$. Then we have enhanced amplitude of the gap soliton and so the number of atoms in GS due to the nonlinear optical lattice. In the center of the gap $\omega = 0$ and the gap soliton solution is

$$U = \frac{\epsilon}{3\gamma_0 \cosh(\epsilon x) + 2\kappa}.$$

When $3\gamma_0 > 2\kappa$,

$$N = \frac{8}{\sqrt{9\gamma_0^2 - 4\kappa^2}} \arctan \frac{(3\gamma_0 - 2\kappa)\gamma}{\sqrt{9\gamma_0^2 - 4\kappa^2}}, \tag{24}$$

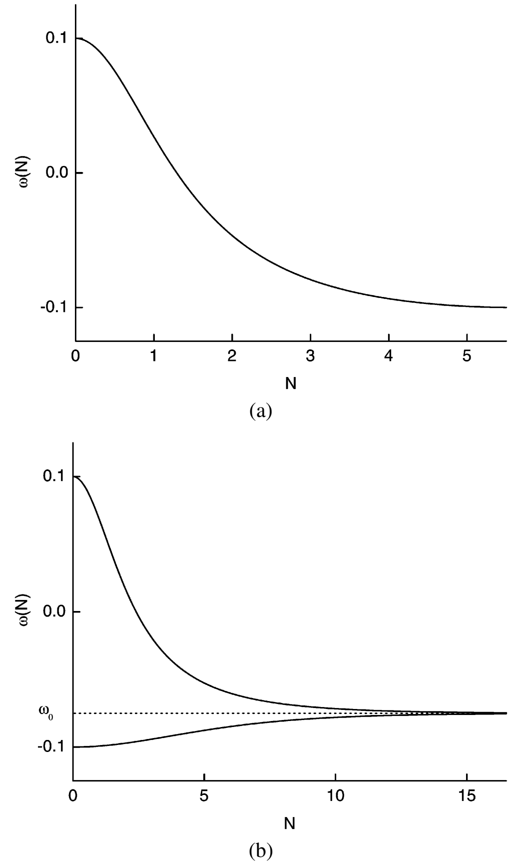

Fig. 2. The chemical potential ω versus norm N. The curve (a) corresponds to condition (27) and the curve (b) to (28).

and when $3\gamma_0 < 2\kappa$,

$$N = \frac{8}{\sqrt{4\kappa^2 - 9\gamma_0^2}} \ln\left|\frac{3\gamma_0 + 2\kappa + \gamma\sqrt{4\kappa^2 - 9\gamma_0^2}}{3\gamma_0 + 2\kappa - \gamma\sqrt{4\kappa^2 - 9\gamma_0^2}}\right|. \tag{25}$$

We can estimate the number of atoms in the gap soliton and the width of the soliton. We have

$$N_s = \frac{\omega_R}{2\omega_\perp \bar{a}_{s0} k} N. \tag{26}$$

For ^{87}Rb atoms in the optical lattice with $\omega = -0.1E_R/\hbar$ and $a_{s1} \sim a_{s0} = 5.4$ nm, $1/k = 0.157$ μm, $V_0 = 0.2E_R$ the number of atoms in soliton is $N_s \approx 10^3$. The soliton width is $w_s \approx 4$ μm.

In the case

$$3\gamma_0 > 2\kappa \tag{27}$$

solution (20) is stable for all values of ω within condition (22), i.e. the soliton exists everywhere in the gap.

In the case $3\gamma_0 < 2\kappa$ for stable solutions we have another condition for ω,

$$-\frac{3\epsilon\gamma_0}{4\kappa} < \omega < \frac{\epsilon}{2}. \tag{28}$$

Chemical potential ω versus norm N is depicted in Fig. 2. According to the Vakhitov–Kolokolov criterion [28] the soliton is stable provided that $dN/d\omega < 0$.

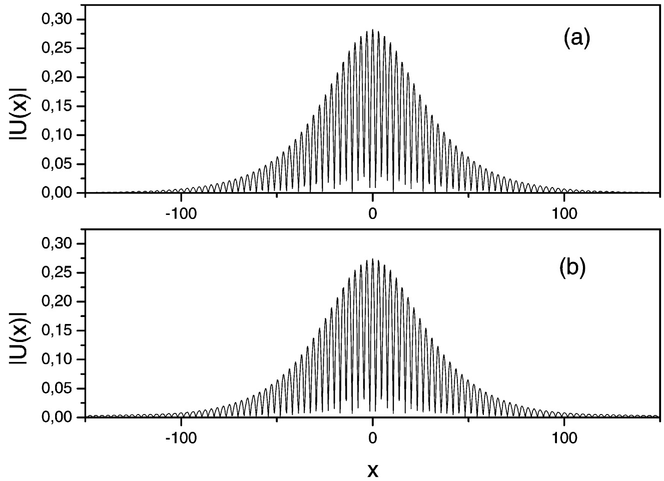

Fig. 3. Stable solution for $\gamma_0 = 1$, $\epsilon = 0.2$, $\kappa = 1$ and $\omega = 0.05$ at different times: (a) $t = 0$ and (b) $t = 100$. The initial wave packet envelope is taken in the form of Eq. (20).

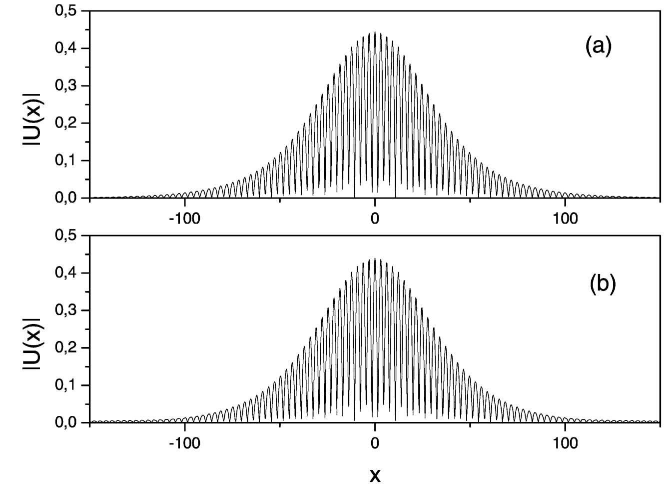

Fig. 4. Stable solution for $\gamma_0 = 0$, $\epsilon = 0.2$, $\kappa = 1$ and $\omega = 0.05$ at different times: (a) $t = 0$ and (b) $t = 100$. The initial wave packet envelope is taken in the form of Eq. (20).

Evolution of the stable solution for $\gamma_0 = 1$, $\epsilon = 0.2$ is given in Fig. 3. The initial wave packet envelope is taken in the form of Eq. (20).

An interesting case we have when $\gamma_0 = 0$, i.e. the homogeneous mean field interaction is equal to zero and the standard gap soliton *is absent*. It can be realized by the detuning the two-body interactions by applying the constant external magnetic field with the Feshbach resonance method. Then the soliton exists in the interval $0 < \omega < \epsilon/2$, i.e. in the upper half of the gap. The gap soliton solution is

$$U(x) = \frac{\epsilon^2 - 4\omega^2}{2\kappa(\epsilon + 2\omega\cosh(2\beta x))}.$$

The number of atoms for $\omega = \Delta \ll \epsilon$ near the center of the gap can be estimated as

$$N \approx \frac{4}{\kappa}\ln\left|\frac{\epsilon}{\Delta}\right|.$$

When $\omega = \epsilon/2 - \Delta$, i.e. the parameter near the band edge, the number of atoms in gap soliton tends to zero like

$$N \to \frac{8}{\kappa}\sqrt{\frac{\Delta}{\epsilon}}.$$

Numerical simulation of this case is shown in Fig. 4.

We performed numerical simulations of the GP equation with the linear–nonlinear optical potentials. In Figs. 3, 4 evolution of the initial state in Eq. (3) is given in the form of Eq. (20).

When $|\tan(\theta/2)| > \gamma$ the expressions for the amplitude U and $\cos\theta$ take the forms

$$\cos\theta = \frac{\tanh^2(\beta x) - \gamma^2}{\tanh^2(\beta x) + \gamma^2}, \tag{29}$$

$$U(x) = -\frac{\epsilon - \omega}{(3\gamma_0 + 2\kappa)\sinh^2(\beta x) + (3\gamma_0 - 2\kappa)\gamma^2\cosh^2(\beta x)}. \tag{30}$$

In this case the solution is evidently unstable. Indeed, in this case $dN/d\omega > 0$ and in accordance with the Vakhitov–Kolokolov criterion [28] such a solution is unstable. PDE simulation of the evolution of an unstable solution for $\gamma_0 = 1$, $\epsilon = 0.2$ and $\kappa = 2$ is given in Fig. 5. The initial wave packet envelope is taken in the form of Eq. (30).

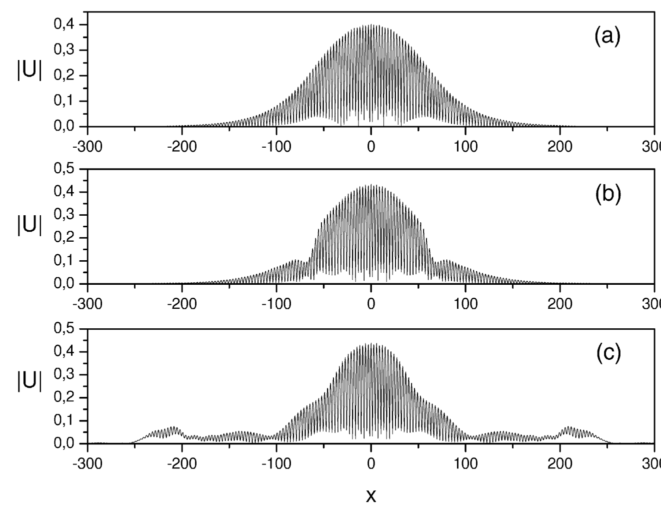

Fig. 5. Unstable solution for $\gamma_0 = 1$, $\epsilon = 0.2$, $\kappa = 2$ and $\omega = -0.08$ at different times: (a) $t = 0$, (b) $t = 40$ and (c) $t = 140$. The initial wave packet envelope is taken in the form of Eq. (30).

4. The case of slowly varying linear optical and rapidly varying nonlinear optical lattices

It is of interest to study the case when the periods of linear and nonlinear lattices are different. Here we restrict ourselves by the case when $L_{nl} \ll L_{lin} = 2\pi$, L_{nl}, L_{lin} are periods of the nonlinear and linear optical lattices, respectively. The GP equation in this case is

$$iu_t + u_{xx} - \epsilon\cos(2x)u + \big(\gamma_0 + \kappa\cos(\mu x)\big)|u|^2 u = 0, \tag{31}$$

where $\mu = 2\pi/L_{nl} \gg 1$. Looking for the solution of the form $u(x,t) = U(x,t) + \mu^{-1}u_1 + \mu^{-2}u_2 + \cdots$, where $U(x,t)$ is the slowly varying field, we obtain the averaged over rapid modulations GP equation

$$iU_t + U_{xx} - \epsilon\cos(2x)U + \gamma_0|U|^2U + \frac{3\kappa^2}{2\mu^2}|U|^4U = 0. \quad (32)$$

We keep terms $\sim O(1/\mu^2)$ when deriving this equation. This equation has the form of the GP equation with optical lattice potential and two- and three-body interactions, that has been studied recently in [29]. The effective three-body interactions term corresponds to attractive interactions. It is distinguished from the analogous equation in nonlinear optical systems, since the quintic term has negative sign (defocusing nonlinearity), corresponding to the expansion of saturable nonlinearity term. If the lattice is switched off, wave packets with number of atoms excessing the critical value

$$N_c = \frac{\pi\mu}{\sqrt{2}\kappa}$$

should collapse. The critical number now depends on the ratio between the wavenumber and amplitude of modulations of the scattering length, i.e. on the strength of the nonlinearity management.

The analysis performed in work [29] (see also [30]) showed that when $\gamma_0 = 0$, in optical lattice should exists the gap-Townes soliton solution. If the number of atoms excess N_c this solution oscillates for many periods and can be observed experimentally in the BEC under linear and nonlinear periodic potentials.

5. Travelling gap-soliton solution. Solitons collision

Travelling solitons can be obtained proceeding from the ansatz of work [19]. The solution of Eqs. (5)–(6) should be sought in the form of the sum of travelling waves with different amplitudes and phases

$$A(x,t) = \sqrt{\Delta}\sqrt{F_A(\xi)}e^{i(\theta_A(\xi)-\omega t)}, \quad (33)$$

$$B(x,t) = \frac{1}{\sqrt{\Delta}}\sqrt{F_B(\xi)}e^{i(\theta_B(\xi)-\omega t)}, \quad (34)$$

where

$$\Delta = \sqrt{\frac{2+v}{2-v}}, \qquad \xi = x - vt.$$

The soliton velocity is varying in the interval $-v_R < v_s < v_R$, where $v_R = \hbar k/m$. Substituting these expressions into Eqs. (5)–(6) we have

$$F_\xi = \frac{\tilde\epsilon}{2}F\sin(\theta_B-\theta_A) - \tilde\kappa F^2\sin(\theta_B-\theta_A), \quad (35)$$

$$(\theta_B-\theta_A)_\xi = -\tilde\omega + \frac{\tilde\epsilon}{2}\cos(\theta_B-\theta_A) - 3\tilde\gamma_0 F - 2\tilde\kappa F\cos(\theta_B-\theta_A), \quad (36)$$

$$(\theta_B+\theta_A)_\xi = \frac{v}{2}\left[\tilde\omega + \frac{8}{4+v^2}\tilde\gamma_0 F + \tilde\kappa F\cos(\theta_B-\theta_A)\right], \quad (37)$$

where $F = F_A = F_B$,

$$\tilde\omega = \lambda^2\omega, \qquad \tilde\epsilon = \lambda\epsilon, \qquad \tilde\kappa = \lambda^2\kappa,$$

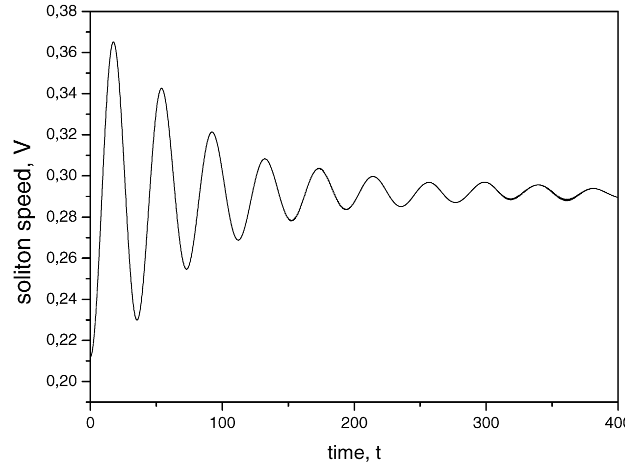

Fig. 6. The time dependence of the travelling soliton speed for $\gamma_0 = 1$, $\epsilon = 0.2$, $\kappa = 1$ and $\omega = 0.05$ when the initial speed $v = 0.2$.

$$\tilde\gamma_0 = \frac{(4+v^2)}{4}\lambda^3\gamma_0, \qquad \lambda = \frac{1}{\sqrt{1-\frac{v^2}{4}}}. \quad (38)$$

The first two equations of this system coincide with Eqs. (13)–(14) with renormalized parameters. Thus the region of the solitons stability are determined again by conditions (22), (28) with renormalized parameters as in Eq. (38).

The gap is given by

$$-\frac{\epsilon}{2}\sqrt{1-\frac{v^2}{4}} < \omega < \frac{\epsilon}{2}\sqrt{1-\frac{v^2}{4}}. \quad (39)$$

When the velocity tends to the limiting value $v \to 2$ (in dimension units to v_R), the gap region shrinks to zero. Also the gap does not exist for the velocity $v > 2$.

The stability condition is

$$-\frac{3(4+v^2)\epsilon\gamma_0}{4\kappa} < \omega < \frac{\epsilon}{2}\sqrt{1-\frac{v^2}{4}}. \quad (40)$$

Numerical simulations with the initial wave packet taken in the form

$$u(x,t) = \left(A(\xi)e^{ix} + B(\xi)e^{ix}\right)e^{-i(\omega+1)t} \quad (41)$$

show that obtained solution of Eq. (3) has the form of an automodel soliton.

In the course of evolution of the initial wave-packet the soliton parameters, the soliton speed, amplitude and width, approach their stationary values. Evolution of the soliton speed is shown in Fig. 6.

To check the nature of obtained moving solutions in Fig. 7 the result of simulation of two solitary waves collision taken in the form of Eq. (41) is shown. As is evident from this figure, collision of two solitons is nearly elastic. A small radiation at the solitons tails seems to be caused by the nonintegrability of the system.

6. Conclusion

In conclusion, we have investigated the dynamics of BEC under joint action of the linear and nonlinear optical lattices.

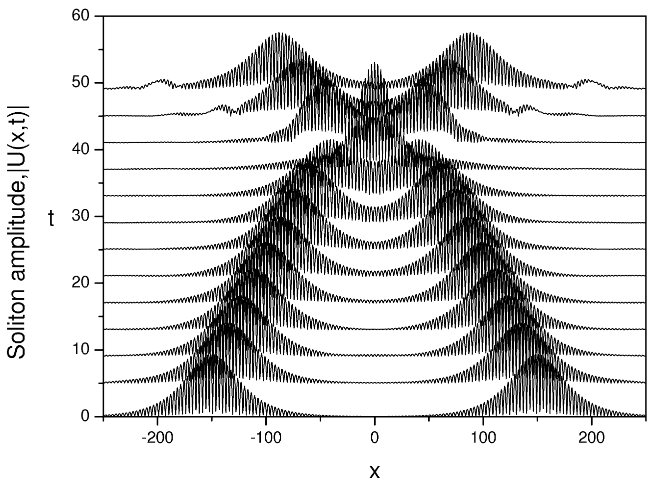

Fig. 7. Collision of two moving solitons for $\gamma_0 = 1$, $\epsilon = 0.2$, $\kappa = 1$ and $\omega = 0.05$.

These lattices can be created by counter propagating laser beams via optically induced Feshbach resonances. The governing equation is the Gross–Pitaevskii equation with the optical potential and periodic nonlinearity. This type of equation also describes the dynamics of the Fermi–Bose mixture in the optical lattice [12] and our analysis is useful for the investigation of gap solitons in such systems. For the case of shallow lattices we have derived the coupled-mode system of equations. We derived the nonlinear dispersion relation for matter waves and analyze the loop behavior in the dispersion curves. New types of gap soliton solutions are found and the soliton stability regions are analyzed. In particular it was observed, that the gap soliton can exist even when the background part of the scattering length is equal to zero. This type of the gap soliton is stable when its parameters are in the upper half of the forbidden gap. When the potential represent a superposition of a slow varying linear and rapidly varying nonlinear optical lattices, the averaged dynamics is described by the GP equation with effective *attractive three-body interactions*. Thus, in such a system at $\gamma_0 = 0$ it becomes possible to observe the gap-Townes soliton predicted in the work [29] and a representative long-lived breathing wave packet.

We also have found the travelling gap soliton solution in BEC and studied collision of gap solitons by numerical simulations. The stability region of moving solitons turned to be shrinks with increasing of the velocity. It has been shown that the collisions of the gap solitons are nearly elastic, with the emitting of small radiation, due to the nonintegrability of the couple-mode system.

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