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MODELING ADIABATIC N-SOLITON INTERACTIONS AND PERTURBATIONS

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We analyze a perturbed version of the complex Toda chain (CTC) in an attempt to describe the adiabatic N-soliton train interactions of the perturbed nonlinear Schrödinger equation. We study perturbations with weak quadratic and periodic external potentials analytically and numerically. The perturbed CTC adequately models the N-soliton train dynamics for both types of potentials. As an application of the developed theory, we consider the dynamics of a train of matter-wave solitons confined in a parabolic trap and an optical lattice.

Keywords: complex Toda chain, adiabatic dynamics, soliton train, expulsion of a soliton

1. Introduction

The N-soliton train interactions for the nonlinear Schrödinger equation (NLS) and its perturbed versions

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = i\varepsilon R[u], \tag{1}$$

whose study started with the pioneering paper [1], have by now been extensively studied (see [2]–[6] and the references therein). Several other nonlinear evolution equations have also been studied, among them some higher NLS equations [6], the Ablowitz–Ladik system [7], and the modified NLS equation [8]–[12].

We concentrate on perturbed NLS equation (1). By an N-soliton train, we mean a solution of the (perturbed) NLS fixed by the initial conditions

$$u(x,t=0) = \sum_{k=1}^{N} u_k^{1s}(x,t=0), \qquad u_k^{1s}(x,t) = \frac{2\nu_k e^{i\phi_k}}{\cosh z_k}, \tag{2}$$

$$z_k(x,t) = 2\nu_k\bigl(x-\xi_k(t)\bigr), \qquad \xi_k(t) = 2\mu_k t + \xi_{k,0}, \tag{3}$$

$$\phi_k(x,t) = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \qquad \delta_k(t) = W_k t + \delta_{k,0}. \tag{4}$$

Each soliton has four parameters: the amplitude ν_k, velocity μ_k, center of mass ξ_k, and phase δ_k. The adiabatic approximation uses the soliton overlap, which falls off exponentially with the distance between the solitons, as a small parameter $\varepsilon_0 \ll 1$. The soliton parameters must then satisfy [1]

$$|\nu_k - \nu_0| \ll \nu_0, \qquad |\mu_k - \mu_0| \ll \mu_0, \qquad |\nu_k - \nu_0|\,|\xi_{k+1,0} - \xi_{k,0}| \gg 1, \tag{5}$$

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where $\nu_0 = (1/N)\sum_{k=1}^{N}\nu_k$ and $\mu_0 = (1/N)\sum_{k=1}^{N}\mu_k$ are the respective average amplitude and velocity. In fact, we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \qquad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \qquad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

It can be expected that the approximation holds only for such times t for which the set of $4N$ parameters of the soliton train satisfy (5).

Equation (1) finds a number of applications in nonlinear optics and is integrable for $R[u] \equiv 0$ via the inverse scattering transform [13], [14]. The N-soliton train dynamics in the adiabatic approximation is modeled by the complex generalization of the Toda chain:

$$\frac{d^2 Q_j}{dt^2} = 16\nu_0^2\left(e^{Q_{j+1}-Q_j} - e^{Q_j - Q_{j-1}}\right), \quad j = 1, \ldots, N. \tag{6}$$

The complex variables Q_k are expressed in terms of the soliton parameters by

$$Q_k(t) = 2i\lambda_0\xi_k(t) + 2k\log(2\nu_0) + i\big(k\pi - \delta_k(t) - \delta_0\big), \tag{7}$$

where $\delta_0 = (1/N)\sum_{k=1}^{N}\delta_k$ and $\lambda_0 = \mu_0 + i\nu_0$. In what follows, we assume free-end conditions, i.e., $e^{-Q_0} \equiv e^{Q_{N+1}} \equiv 0$.

We note that the N-soliton train is *not* an N-soliton solution. The spectral data of the corresponding Lax operator L is also nontrivial on the continuous spectrum of L. Therefore the analytic results from the soliton theory cannot be applied. Moreover, we want to treat solitons moving with equal velocities and also to account for the effects of possible nonintegrable perturbations $R[u]$.

This paper extends the results in [2]–[5], [12], [15]. Recently, in connection with the realization of Bose–Einstein condensation in dilute atomic gases, it became important to study the NLS equation with an additional potential term $iR[u] = V(x)u(x,t)$ (see [16], [17]). We continue the analysis of the corresponding perturbed CTC (PCTC) model for quadratic and periodic potentials $V(x)$ in [6]. Our results confirm the stabilization properties of the periodic potentials observed in [18], [19] in a different physical setup.

2. The importance of the CTC model

Because the CTC, like the (real) Toda chain (RTC), is a completely integrable Hamiltonian system [20], we can analyze the asymptotic behavior of the N-soliton trains analytically. But unlike the RTC, the CTC has a rich variety of dynamical regimes [21], such as

- asymptotically free motion if $v_j \neq v_k$ for $j \neq k$ (this is the only possible dynamical regime for the RTC),
- an N-soliton bound state if $v_1 = \cdots = v_N$ but $\zeta_k \neq \zeta_j$ for $k \neq j$,
- various intermediate (mixed) regimes (e.g., if $v_1 = v_2 > \cdots > v_N$ but $\zeta_k \neq \zeta_j$ for $k \neq j$, then we have a bound state of the first two solitons while all the others are asymptotically free), and
- singular and degenerate regimes if two or more of the eigenvalues of L become equal (e.g., $\zeta_1 = \zeta_2$ and $\zeta_j \neq \zeta_k$ for $2 < j \neq k$).

Above, the $\zeta_k = v_k + iw_k$ denote the eigenvalues of the Lax matrix L in the Lax representation $L_\tau = [M, L]$ of the CTC, where

$$\begin{gathered} L = \sum_{k=1}^{N} b_k E_{kk} + \sum_{k=1}^{N-1} a_k(E_{k,k+1} + E_{k+1,k}), \\ b_k \equiv -\frac{1}{2}\frac{dQ_k}{d\tau} = \frac{1}{2}(\mu_k + i\nu_k), \qquad a_k = \frac{1}{2}\exp\frac{Q_{k+1} - Q_k}{2}, \end{gathered} \tag{8}$$

and the matrices $(E_{kp})_{ij} = \delta_{ki}\delta_{pj}$. The eigenvalues ζ_k of L are time-independent and complex along with the first components of the normalized eigenvectors of L:

$$L\vec{z}^{(k)} = \zeta_k \vec{z}^{(k)}, \qquad \left(\vec{z}^{(k)}, \vec{z}^{(m)}\right) = \delta_{km}. \tag{9}$$

The set $\{\zeta_k = v_k + iw_k,\ \eta_k = \sigma_k + i\theta_k\}$ can be viewed as the set of action–angle variables of the CTC.

Using the CTC model, we can determine the asymptotic regime of the N-soliton train. Given the initial parameters $\mu_k(0)$, $\nu_k(0)$, $\xi_k(0)$, and $\delta_k(0)$ of the N-soliton train, we can calculate the matrix elements b_k and a_k of L at $t = 0$. Solving the characteristic equation on $L|_{t=0}$, we can calculate the eigenvalues ζ_k to determine the asymptotic regime of the N-soliton train [2], [5]. Another option is to impose a specific constraint on ζ_k, for example, that all ζ_k are purely imaginary, i.e., all $v_k = 0$. This provides a set of algebraic conditions $L|_{t=0}$ and conditions on the initial soliton parameters $\mu_k(0)$, $\nu_k(0)$, $\xi_k(0)$, and $\delta_k(0)$, which characterize the region in the soliton parameter space responsible for the N-soliton bound states.

3. The perturbed NLS and PCTC: Quadratic and periodic potentials

We consider several specific choices $R^{(p)}[u]$, $p = 1, 2, \ldots$, of perturbations in (1). In the adiabatic approximation, the dynamics of the soliton parameters can be determined by the system (see [1] for $N = 2$ and [2], [5] for $N > 2$)

$$\frac{d\lambda_k}{dt} = -4\nu_0\left(e^{Q_{k+1}-Q_k} - e^{Q_k - Q_{k-1}}\right) + M_k^{(p)} + iN_k^{(p)}, \tag{10}$$

$$\frac{d\xi_k}{dt} = 2\mu_k + \Xi_k^{(p)}, \qquad \frac{d\delta_k}{dt} = 2(\mu_k^2 + \nu_k^2) + X_k^{(p)}, \tag{11}$$

where $\lambda_k = \mu_k + i\nu_k$ and $X_k^{(p)} = 2\mu_k \Xi_k^{(p)} + D_k^{(p)}$. The right-hand sides of Eqs. (10) and (11) are determined by $R_k^{(p)}[u]$ via

$$N_k^{(p)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k}{\cosh z_k} \operatorname{Re}\left(R_k^{(p)}[u]e^{-i\phi_k}\right), \tag{12}$$

$$M_k^{(p)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k \sinh z_k}{\cosh^2 z_k} \operatorname{Im}\left(R_k^{(p)}[u]e^{-i\phi_k}\right), \tag{13}$$

$$\Xi_k^{(p)} = \frac{1}{4\nu_k^2}\int_{-\infty}^{\infty} \frac{dz_k\, z_k}{\cosh z_k} \operatorname{Re}\left(R_k^{(p)}[u]e^{-i\phi_k}\right), \tag{14}$$

$$D_k^{(p)} = \frac{1}{2\nu_k}\int_{-\infty}^{\infty} \frac{dz_k\,(1 - z_k \tanh z_k)}{\cosh z_k} \operatorname{Im}\left(R_k^{(p)}[u]e^{-i\phi_k}\right). \tag{15}$$

Substituting (10) and (11) in (7), we derive

$$\begin{gathered}
\frac{dQ_k}{dt} = -4\nu_0\lambda_k + \frac{2k}{\nu_0}\mathcal{N}_0^{(p)} + 2i\xi_k\left(\mathcal{M}_0^{(p)} + i\mathcal{N}_0^{(p)}\right) + i\left(2\lambda_0\Xi_k^{(p)} - X_k^{(p)} - \mathcal{X}_0^{(p)}\right), \\
\mathcal{N}_0^{(p)} = \frac{1}{N}\sum_{j=1}^{N} N_j^{(p)}, \qquad \mathcal{M}_0^{(p)} = \frac{1}{N}\sum_{j=1}^{N} M_j^{(p)}, \qquad \mathcal{X}_0^{(p)} = \frac{1}{N}\sum_{j=1}^{N} X_j^{(p)}.
\end{gathered} \tag{16}$$

In deriving Eq. (16), we keep terms of the order $\Delta\nu_k \simeq \mathcal{O}\left(\sqrt{\varepsilon_0}\right)$ and neglect terms of the order $\mathcal{O}(\varepsilon_0)$. As a result of the perturbations, ν_0 and μ_0 may become time-dependent. Indeed, from (10), we obtain

$$\frac{d\mu_0}{dt} = \mathcal{M}_0^{(p)}, \qquad \frac{d\nu_0}{dt} = \mathcal{N}_0^{(p)}. \tag{17}$$

The small parameter ε_0 can be related to the initial distance $r_0 = |\xi_2 - \xi_1|_{t=0}$ between the two solitons. Assuming that $\nu_{1,2} \simeq \nu_0$, we obtain

$$\varepsilon_0 = \int_{-\infty}^{\infty} dx\, \left| u_1^{1\mathrm{s}}(x,0) u_2^{1\mathrm{s}}(x,0) \right| \simeq 8\nu_0 r_0 e^{-2\nu_0 r_0}. \tag{18}$$

In particular, (18) means that $\varepsilon_0 \simeq 0.01$ for $r_0 \simeq 8$ and $\nu_0 = 1/2$.

We assume that the solitons are initially ordered such that $\xi_{k+1} - \xi_k \simeq r_0$. It can be verified [3], [12] that $N_k^{(p)} \simeq M_k^{(p)} \simeq \exp\bigl(-2\nu_0 |k-p| r_0\bigr)$. Therefore, the interaction terms between the kth and $(k\pm1)$th solitons are of the order of $e^{-2\nu_0 r_0}$; the interactions between the kth and $(k\pm2)$th soliton are of the order of $e^{-4\nu_0 r_0} \ll e^{-2\nu_0 r_0}$.

The terms $\Xi_k^{(0)}$ and $X_k^{(0)}$ are of the order of $r_0^a \exp(-2\nu_0 r_0)$, where $a = 0$ or 1. But they can be neglected as compared to $\tilde{\mu}_k$ and $\tilde{\nu}_k$, where

$$\tilde{\mu}_k = \mu_k - \mu_0 \simeq \sqrt{\varepsilon_0}\,, \qquad \tilde{\nu}_k = \nu_k - \nu_0 \simeq \sqrt{\varepsilon_0}\,. \tag{19}$$

The corrections to $N_k^{(p)}, \ldots$, coming from the terms linear in u depend only on the parameters of the kth soliton, i.e., they are "local" in k. The nonlinear terms in u present in $iR^{(p)}[u]$ also produce terms in $N_k^{(p)}, \ldots$ that are "nonlocal" in k.

We now consider perturbations of the form $iR[u] = V(x)u(x,t)$. Our first choice for $V(x)$ is a quadratic potential:

$$V^{(1)}(x) = V_2 x^2 + V_1 x + V_0. \tag{20}$$

Skipping the details, we present the results [6]:

$$N_k^{(1)} = 0, \qquad M_k^{(1)} = -V_2 \xi_k - \frac{V_1}{2}, \tag{21a}$$

$$\Xi_k^{(1)} = 0, \qquad D_k^{(1)} = V_2 \left(\frac{\pi^2}{48\nu_k^2} - \xi_k^2 \right) - V_1 \xi_k - V_0, \tag{21b}$$

and $X_k^{(1)} = D_k^{(1)}$. Another important choice is the periodic potential

$$V^{(2)}(x) = A \cos(\Omega x + \Omega_0), \tag{22}$$

where A, Ω, and Ω_0 are appropriately chosen constants. The NLS equation with similar potentials appears naturally in the study of Bose–Einstein condensates (BECs) [16]. For two interacting solitons, the corresponding Karpman–Solov'ev system was derived in [19]. For $N > 2$, we obtain the PCTC, where the integrals for N_k, M_k, Ξ_k, and D_k are equal to [6]

$$N_k^{(2)} = 0, \qquad M_k^{(2)} = \frac{\pi A \Omega^2}{8\nu_k} \frac{1}{\sinh Z_k} \sin(\Omega \xi_k + \Omega_0), \tag{23}$$

$$\Xi_k^{(2)} = 0, \qquad D_k^{(2)} = -\frac{\pi^2 A \Omega^2}{16\nu_k^2} \frac{\cosh Z_k}{\sinh^2 Z_k} \cos(\Omega \xi_k + \Omega_0), \tag{24}$$

where $Z_k = \pi\Omega/(4\nu_k)$. These results allow deriving the corresponding PCTC models.

4. Analysis of the PCTC and comparison with numerical simulations

The dynamics of an individual soliton in a train is determined by the combined action of the external potential and the influence of neighboring solitons. The interaction with neighboring solitons can be either repulsive or attractive, depending on the phase relations between them. In particular, if their amplitudes are equal and the initial phase difference between the neighboring solitons is π (as considered below), they repel each other, resulting in the soliton train expanding in the absence of an external field [2], [3].

The external potential counterbalances the expansion, trying to confine solitons in the potential minima. The interplay of these two factors, the interaction between solitons and the action of the external potential, results in the rich dynamics of the N-soliton train.

To verify the adequacy of the PCTC model for describing the dynamics of the N-soliton train in external potentials, we compared predictions of the corresponding PCTC system and direct simulations of the underlying NLS equation (1). Below, we present results pertaining to a matter-wave soliton train in a confining parabolic trap and in a periodic potential modeling an optical lattice.

4.1. Quadratic potential. We consider the dynamics of an N-soliton train confined to a quadratic potential $V(x) = V_2x^2 + V_1x + V_0$. This application is particularly interesting because it is directly relevant to recent experiments with a train of BEC solitons [22]. Theoretical analysis of the BEC train in periodic traps involving the Toda lattice approach was presented in [17]. The corresponding PCTC system in terms of soliton parameters has the form

$$\frac{d\mu_k}{dt} = 16\nu_0^3\bigl(e^{-2\nu_0(\xi_{k+1}-\xi_k)}\cos\bigl(2\mu_0(\xi_{k+1}-\xi_k)+\delta_k-\delta_{k+1}\bigr) - - e^{-2\nu_0(\xi_k-\xi_{k-1})}\cos\bigl(2\mu_0(\xi_k-\xi_{k-1})+\delta_{k-1}-\delta_k\bigr)\bigr) - V_2\xi_k - \frac{V_1}{2}, \tag{25}$$

$$\frac{d\nu_k}{dt} = 16\nu_0^3\bigl(e^{-2\nu_0(\xi_{k+1}-\xi_k)}\sin\bigl(2\mu_0(\xi_{k+1}-\xi_k)+\delta_k-\delta_{k+1}\bigr) - - e^{-2\nu_0(\xi_k-\xi_{k-1})}\sin\bigl(2\mu_0(\xi_k-\xi_{k-1})+\delta_{k-1}-\delta_k\bigr)\bigr), \tag{26}$$

$$\frac{d\xi_k}{dt} = 2\mu_k, \tag{27}$$

$$\frac{d\delta_k}{dt} = 2(\mu_k^2+\nu_k^2) + V_2\left(\frac{\pi^2}{48\nu_k^2} - \xi_k^2\right) - V_1\xi_k - V_0, \tag{28}$$

where μ_k is the velocity, ν_k is the amplitude, ξ_k is the center of mass, and δ_k is the phase of the kth soliton and where μ_0 and ν_0 are the average velocity and amplitude of the train.

It is reasonable to assume that $V_2 \simeq \mathcal{O}(\varepsilon_0/N)$, which ensures the possibility that the N-soliton train is "inside" the potential. This also means that the exponential terms and the correction terms $M_k^{(1)}$ have the same order of magnitude. Equation (26) then gives $d\nu_0/dt = 0$, and from Eqs. (25) and (27), we obtain

$$\frac{d\mu_0}{dt} = -V_2\xi_0 - \frac{V_1}{2}, \qquad \frac{d\xi_0}{dt} = 2\mu_0, \tag{29}$$

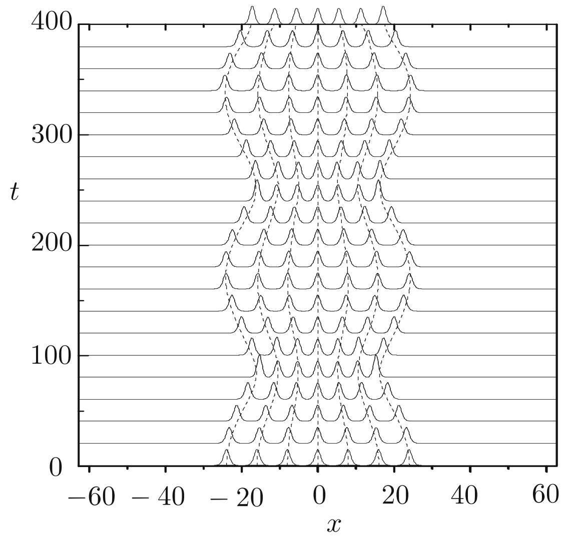

Fig. 1. A seven-soliton train in a parabolic trapping potential $V(x) = 0.0001x^2$ periodically contracts and expands around the center of the trap: solid lines are the results of directly simulating NLS equation (1) numerically, and dashed lines are the result of solving PCTC system (25)–(28) for the center of mass ξ_i. The initial phase difference and space between neighboring solitons are respectively π and 8. The solitons have equal amplitudes $2\nu = 1$.

where μ_0 is the average velocity and $\xi_0 = (1/N)\sum_{j=1}^{N}\xi_j$ is the center of mass of the N-soliton train. System of equations (29) for $V_2 > 0$ has the simple solution

$$\mu_0(t) = \mu_{00}\cos\bigl(\Phi(t)\bigr), \qquad \xi_0(t) = \sqrt{\frac{2}{V_2}}\,\mu_{00}\sin\bigl(\Phi(t)\bigr) - \frac{V_1}{2V_2}, \tag{30}$$

where $\Phi(t) = \sqrt{2V_2}\,t + \Phi_0$ and μ_{00} and Φ_0 are integration constants. From Eq. (29), we find that $d\nu_0/dt = 0$. Therefore, the overall effect of such a quadratic potential is to induce a slow periodic motion of the train as a whole.

If the center of mass of the N-soliton train coincides with the minimum of the quadratic potential, then the soliton train periodically contracts and expands, as shown in Fig. 1.

When the N-soliton train is placed off-center in the parabolic trap, the dynamics is more complicated. The train center of mass oscillates harmonically around the potential minimum, combined with contraction and expansion of the soliton train, as shown in Fig. 2. It follows from Eq. (30) that the period of the center-of-mass motion is $T = 2\pi/\sqrt{2V_2}$; for the parameters in Fig. 2, we have $T = 314.2$, which agrees well with the numerical simulations.

The direct simulations of NLS equation (1) show that a stronger parabolic trap may cause individual solitons to merge at the times of contraction, the original configuration being restored when the train expands. This behavior is reminiscent of the "missing soliton" phenomenon observed in the experiment [22]. But this situation is beyond the validity range of the PCTC approach.

4.2. Periodic potential. Another external potential in which the N-soliton train exhibits interesting dynamics is the periodic potential of the form $V(x) = A\cos(\Omega x + \Omega_0)$. This case may also be directly relevant to matter-wave soliton trains confined to optical lattices. The PCTC system in terms of the soliton

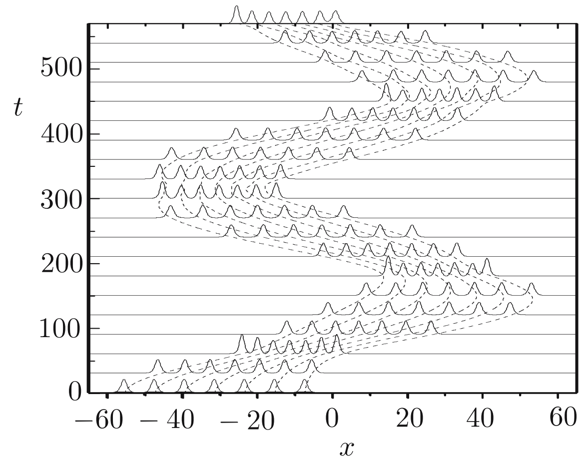

Fig. 2. Dynamics of a seven-soliton train placed off-center in the trap $V(x) = 0.0002x^2$: solid lines are the results of directly simulating NLS equation (1) numerically, and dashed lines are the result of solving PCTC system (25)–(28) for the center of mass ξ_i. The parameters of solitons are the same as in Fig. 1. The initial displacement of the soliton train is -10π.

parameters has the form

$$\frac{d\mu_k}{dt} = 16\nu_0^3\big(e^{-2\nu_0(\xi_{k+1}-\xi_k)}\cos\big(2\mu_0(\xi_{k+1}-\xi_k)+\delta_k-\delta_{k+1}\big) - \\ - e^{-2\nu_0(\xi_k-\xi_{k-1})}\cos\big(2\mu_0(\xi_k-\xi_{k-1})+\delta_{k-1}-\delta_k\big)\big) + M_k^{(2)}(\nu_k), \tag{31}$$

$$\frac{d\nu_k}{dt} = 16\nu_0^3\big(e^{-2\nu_0(\xi_{k+1}-\xi_k)}\sin\big(2\mu_0(\xi_{k+1}-\xi_k)+\delta_k-\delta_{k+1}\big) - \\ - e^{-2\nu_0(\xi_k-\xi_{k-1})}\sin\big(2\mu_0(\xi_k-\xi_{k-1})+\delta_{k-1}-\delta_k\big)\big), \tag{32}$$

$$\frac{d\xi_k}{dt} = 2\mu_k, \tag{33}$$

$$\frac{d\delta_k}{dt} = 2(\mu_k^2+\nu_k^2) + D_k^{(2)}(\nu_k), \tag{34}$$

where $M_k^{(2)}(\nu_k)$ and $D_k^{(2)}(\nu_k)$ are given in (23) and (24).

Each soliton of the train experiences the confining force of the periodic potential and the repelling force of neighboring solitons. Therefore, equilibrium positions of solitons do not coincide with the minima of the periodic potential. Solitons placed initially at minima of the periodic potential (Fig. 3) perform small-amplitude oscillations around these minima, provided that the potential is sufficiently strong to keep the solitons confined. In contrast, a weak periodic potential is unable to confine the solitons, and the repulsive forces between neighboring solitons (at the phase difference π) induces an unbounded expansion of the train.

In the intermediate region, when the confining force of the periodic potential is comparable to the repelling forces of neighboring solitons, interesting dynamics can be observed, such as the expulsion of border solitons from the train, as shown in Fig. 4. This phenomenon, revealing the complexity of the internal dynamics of the train, can be explained as follows. Each soliton oscillates nonlinearly within its individual potential well under repulsive forces from neighboring solitons. When the oscillation amplitude

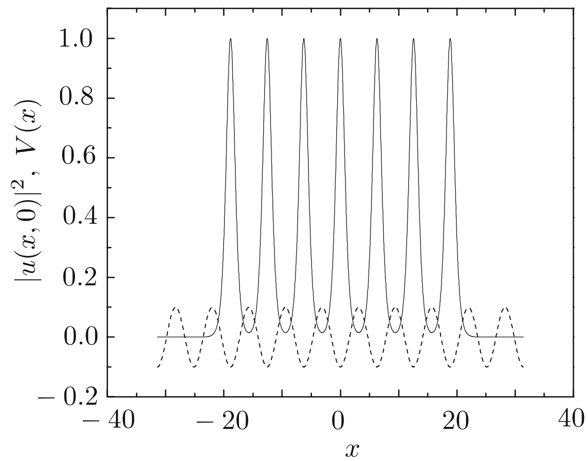

Fig. 3. Solitons (continuous line) remain confined around the minima of the periodic potential $V(x) = A\cos(x)$ (dashed line) performing small-amplitude oscillations if the potential is sufficiently strong $A = -0.1$.

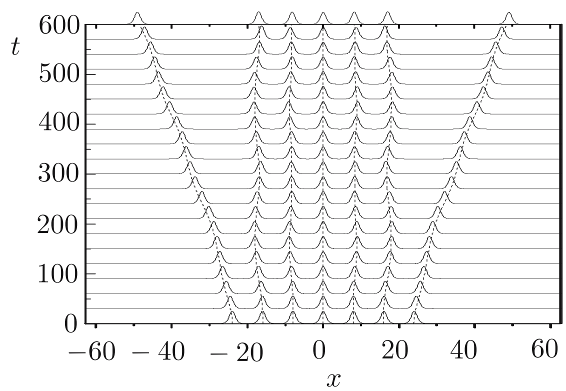

Fig. 4. The expulsion of solitons from the train, as obtained by directly simulating NLS equation (1) (solid lines) and as predicted by PCTC system (31)–(34) for the center of mass ξ_i (dashed lines). The solitons have equal amplitudes $2\nu = 1$ and have the phase difference π and separation of 8 between neighbors. The periodic potential is chosen as $V(x) = -0.001\cos(\pi x/4)$.

of particular solitons increases and two solitons closely approach each other, a strong recoil momentum can cause the soliton to leave the train, overcoming barriers of the periodic potential. In Fig. 4, this happens with border solitons (the other solitons remain bounded under long time evolution). We stress that this phenomenon is well described by the PCTC model, as is evident from Fig. 4.

5. Conclusions

We have studied the dynamics of the N-soliton train confined to external fields (quadratic and periodic potentials). Both the analytic treatment in the framework of the PCTC model and numerical analysis by directly simulating the underlying NLS equation show that the PCTC is adequate for describing the N-soliton interactions in external potentials. We briefly mentioned the relevance of this study to the research on matter-wave soliton trains in magnetic traps and optical lattices.

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