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PHYSICS LETTERS A
Physics Letters A 345 (2005) 237–244
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Coupled-mode theory for Bose–Einstein condensates with time dependent atomic scattering length

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Received 14 June 2005; accepted 6 July 2005

Available online 14 July 2005

Communicated by V.M. Agranovich

Abstract

We study nonlinear tunnelling and localization phenomena in Bose–Einstein condensate in a double-well potential, using a coupled-mode theory. We consider the effects induced by the damping and an atomic scattering length oscillating in time in a double-well potential. The existence of synchronous solutions in a running phase mode and resonances in macroscopic quantum tunnelling for such system are investigated. The switching between macroscopic quantum tunnelling and quantum self-trapped regimes induced by periodic modulations of the scattering length is also studied.

PACS: 03.75.Fi; 05.30.Jp

Keywords: Bose–Einstein condensate; Double-well trap; Nonlinear tunnelling; Quantum self-trapping; Oscillating scattering length

1. Introduction

The interaction between two overlapped Bose–Einstein condensates (BEC) induces effects similar to the Josephson oscillations phenomena. The macroscopic quantum tunnelling (MQT) and quantum self-trapping (QST) effects are predicted for BEC in weakly coupled wells [1–3,5,6]. Two-mode system considered in these works is equivalent to the nonlinear dimer model, which appears in many problem of physics: e.g., involving a nonlinear directional coupler in optics [7], condensed matter physics. Recently the MQT and QST phenomena have been observed in the experiment with BEC in a two well trap [8].

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.07.011

The periodic variations in time of the trap potential and the atomics scattering length lead to the resonances in oscillations of a fractional atomic population as in the tunnelling in the self-trapping regimes [9–12]. Also we should note that the damping induces the instability in MQT and MST regimes and leads to the switching from one dynamical state to other. These effects open the possibility to control the nonlinear tunnelling phenomena, by a resonant enhancing or suppression of the tunnelling. These results are obtained in the case of weakly overlapped condensates.

The periodic modulation of trap or atomic scattering length can stabilize the dynamical states in the presence of damping. It is of interest to investigate this possibility and also the existence and properties of resonances in macroscopic quantum tunnelling *beyond* the weak coupling limit. The analysis of influence of the damping on macroscopic quantum tunnelling and quantum self-trapped regimes is also important for the application of the theory to real systems.

This work is devoted to the investigation of this problem. We apply the coupled-mode theory developed for the condensate in the static double-well trap in the article [13]. In this approach the nonlinear collective modes corresponding to the ground and higher-order states are considered. The nonlinearity induces modes coupling and the exchange of atoms between modes. The approach involves a coupling between the ground and first excited (antisymmetric) modes. The model is valid at any separations between wells and describe smoothly two different regimes, occurring when the distance is varied: for a large separation—the Josephson oscillations for weakly coupled BEC, and for a close separation—a nonlinear oscillations of the atomic imbalance between modes, analogous to the Rabi oscillations

We will use this approach to study the influence of damping and periodic modulations in time of the atomic scattering length on tunnelling processes in the double-well potential.

2. Formulation of problem: Varying in time scattering length

The dynamics of BEC in the cigar type trap with quasi 1D dynamics is described by the 1D Gross–Pitaevskii equation [14]

$$i\hbar u_t + \frac{\hbar^2}{2m}u_{xx} - U(x)u + g_{1\mathrm{D}}|u|^2u = 0, \tag{1}$$

where the effective 1D interaction constant is $g_{1\mathrm{D}} = g_{3\mathrm{D}}/(2\pi a_{\perp}^2)$, with the transverse length of oscillator $a_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$. The constant $g_{3\mathrm{D}} = 4\pi\hbar^2 a_s/m$, where a_s is the atomic scattering length. Thus $g_{1\mathrm{D}} = 2a_s\hbar\omega_{\perp}$. It is useful to introduce dimensionless variables according to

$$x = \frac{x}{a_{\perp}}, \qquad t = t\omega_{\perp}, \qquad \psi = \sqrt{2|a_s|}u. \tag{2}$$

Then the problem of the dynamics of BEC in a double-well trap $U(x,t)$ can be described by the next model

$$i\psi_t + \psi_{xx} - U(x,t)\psi + \sigma\bigl(1 + f(t)\bigr)|\psi|^2\psi = 0, \tag{3}$$

where $\sigma = \sigma_0 + \sigma_1(t)$, $\sigma_0 = -\mathrm{sign}(a_s) = \pm 1$ is the taken with the minus sign scattering length. The norm of ψ is $\bar{N} = \int |\psi|^2\,dx = Na_s/a_{\perp}$, N is the number of atoms in the condensate. The trap potential is assumed to be

$$U(x,t) = \kappa\bigl(|x| - x_0\bigr)^2.$$

Below we will consider the case of variation in time of the atomic scattering length $a_s = a_s(t) = a_0(1 + f(t))$, $f(t) = f\sin(\omega t)$. Such variation can be obtained, for example, by the periodic variation in time of the transverse frequency $\omega_{\perp}$ or by the Feshbach resonance techniques [15]. In the latter case the variation can be obtained by changing an external magnetic field, since near the resonance we have:

$$a_s(t) = a_B\left(1 + \frac{D}{B_0 - B(t)}\right), \tag{4}$$

where a_B is the asymptotic value of a_s and $B(t)$ is the variable external magnetic field.

Let $\psi = \Phi_j \exp(-i\beta_j t)$, where Φ_j are the eigenfunctions of the nonlinear problem

$$\Phi_{jxx} + \beta_j\Phi_j - U_0(x)\Phi_j + \sigma\Phi_j^3 = 0. \tag{5}$$

Family of the solutions Φ_j is characterized by parameters β_j and $N_j = \int |\Phi_j|^2\,dx$.

It is useful to use a decomposition for the solution

$$\psi_j(x,t) = b_0(t)\Phi_0(x)e^{-i\beta_0 t} + b_1\Phi_1(x)e^{-i\beta_1 t}. \tag{6}$$

Substituting Eq. (6) into the Gross–Pitaevskii equation (3) and multiplying on Φ_j and integrating over the spatial variable we obtain the system of coupled-mode equations for b_j [13]

$$iB_{0t} + \sigma\big(C_0|B_0|^2B_0 + 2C_{01}|B_1|^2B_0 + C_{01}B_1^2B_0^*e^{-i\Omega t}\big) = 0, \tag{7}$$

$$iB_{1t} + \sigma\big(C_1|B_1|^2B_1 + 2C_{01}B_1|B_0|^2 + C_{01}B_0^2B_1^*e^{i\Omega t}\big) = 0, \tag{8}$$

where

$$\Omega = 2(\beta_1 - \beta_0) - 2\sigma(C_0N_0 - C_1N_1),$$

and $C_i = \int \Phi_i^2\,dx/N_i^2$, $C_{01} = \int \Phi_0^2\Phi_1^2/(N_0N_1)$.

Introducing the new variables: the relative atomic population $\Delta(t) = n_1 - n_0$, and relative phase $\theta = \phi_1 - \phi_0$ we find the system [13]

$$\Delta_t = -\sigma C_{01}\big(n^2 - \Delta^2\big)\sin\theta + R_i, \tag{9}$$

$$\theta_t = \delta - \sigma(C_0 + C_1)\Delta + 2\sigma C_{01}\big(2 + \cos(\theta)\big)\Delta, \quad i = 1, 2, \tag{10}$$

where $\delta = 2(\beta_1 - \beta_0) + \sigma[(n - 2N_0)C_0 - (n - 2N_1)C_1]$. We include into this system the term corresponding to the linear damping effect—terms $R_1 = -\eta\Delta$ and $R_2 = -\epsilon\theta_t$ in the equation for Δ. The first term corresponds to the phenomenological linear damping term in the rhs of the GP equation (3). Such terms also appear in the two-mode model in a two-component BEC and correspond to finite life of modes. The second type of damping corresponds to the normal current contribution which is proportional to the chemical potential difference [3]. The investigation of the validity of two modes expansion represent a separate problem. In the case of weak nonlinearity the perturbation theory can be developed to find the eigenmodes [4]. According to these results, assuming $B_{0,1} \gg B_m, m > 1$ we have for example for the symmetric solution the estimate

$$B_{2k} \sim \sigma\frac{\gamma_{2k,0,0,0}}{\Omega - \Omega_{2k}}|B_0|^2B_0, \quad \gamma_{m,n,l,k} = \int_{-\infty}^{\infty}\Phi_m\Phi_n\Phi_l\Phi_k\,dx,$$

where $\Omega \approx \Omega_0 + \sigma\gamma_{0,0,0,0}|B_0|^2$, and Ω_0 is the eigenvalue of the double-well trap potential. Thus due to the smallness of the nonlinearity and the overlap integrals between modes the contribution of higher modes can be in the first approximation considered as small.

The periodic modulation of the parameters of system can induce a transition between modes and resulting in higher modes being excited. Thus the two-mode model is destroyed. To avoid this the frequency of modulations should satisfy the restriction $\omega_\Delta \ll \Delta\beta$, where ω_Δ is the frequency of atomic imbalance in double-well potential and $\Delta\beta$ is the difference of the chemical potentials. As we see below $\omega_\Delta \sim C_{01}n$, n is the number of atoms, so we have the restriction $C_{01} \leqslant \Delta\beta/n$. This condition is very much filled in this Letter. For weak nonlinearity case the

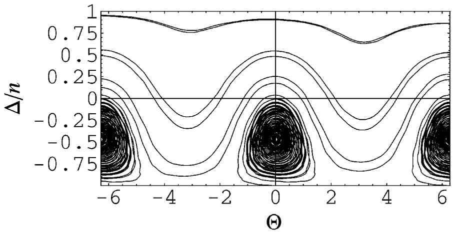

Fig. 1. Phase portrait of the system $\Delta(\Theta)$ with the use of Eqs. (9) and (10) for $x_0 = 1.7$.

chemical potential can be approximated as $\delta\beta = \delta\beta_0 + C_0\sigma N$. Thus the frequency detuning $\Delta\omega$ from the linear problem frequency Ω_0 should satisfy to the condition $\Delta\omega < C_0\sigma N$.

Should be also noted that under the harmonic perturbation of a small amplitude the system still evolving in the limited region of energy. Addition of higher modes which leads to the extension of the energy space as show the analysis of the work [11] has a little influence on the system.

The fixed points of the system (9) at $\eta = 0$ and the constant $\sigma = \sigma_0$ are:

$$\Delta_c = \frac{\sigma_0\delta}{C_0 + C_1 - 6C_{01}}, \quad \theta = 0, 2n\pi, \ n = 0, 1, 2, \ldots . \tag{11}$$

For the large separation between wells $x_0 > 3, C_0 \sim C_1 \sim C_{01}$ and we get $\Delta_c \sim \sigma/(2C_{01}) \approx -(\beta_1 - \beta_0)/(2C_{01})$. Linearizing Eqs. (9) near the fixed point by $\Delta = \Delta_c + \Delta_1, \theta = \theta_0 + \theta_1$ we obtain the frequency of small oscillations near the fixed point:

$$\omega_\Delta^2 = C_{01}\left(n^2 - \Delta_c^2\right)(6C_{01} - C_0 - C_1). \tag{12}$$

In Fig. 1 we plot the phase portrait of the system for parameters $x_0 = 1.7$. Note that our theory is developed for $\beta_1 - \beta_0 \neq 0$. If $\beta_1 - \beta_0 = 0$, then the periodic variation can be included into a new time variable $\tau = \int_0^t \sigma(t')\,dt'$.

3. Dynamics in the running phase regime

Let us consider the dynamics in the case when the phase is unbounded, i.e., the phase is running. The mechanical analog of this state is the steady closed-loop rotations of a nonrigid pendulum around its support point. The influence of the damping on this state is simple—the relative imbalance is diminishing and as a result the system goes down to the point $\Delta = 0$. The parametric drive can compensate the damping effect. We now find the condition for synchronous oscillations in the rotating phase regime. Introducing the slow variable $\phi = \theta - \omega t$ and averaging equations over the period of fast oscillations $2\pi/\omega$ we find the system of equations for slowly varying $\bar{\Delta}, \phi$

$$\begin{aligned} \bar{\Delta}_t &= -\frac{1}{2}\sigma_0 f C_{01}\left(n^2 - \bar{\Delta}^2\right)\cos(\phi) - \eta\bar{\Delta}, \\ \phi_t &= -\omega + \delta_0 - \sigma_0(C_0 + C_1)\bar{\Delta} + 4\sigma_0 C_{01}\bar{\Delta} - \sigma f C_{01}\sin(\phi)\bar{\Delta}. \end{aligned} \tag{13}$$

Taking derivatives Δ_t, ϕ_t equal to zero we can find the fixed points:

$$\bar{\Delta}_c = \frac{\sigma_0(\delta_0 - \omega)}{(C_0 + C_1 - 4C_{01} + fC_{01}\sin(\phi_c))}, \tag{14}$$

$$\cos(\phi_c) = -\frac{2\sigma_0\eta\bar{\Delta}_c}{C_{01}f(n^2 - \bar{\Delta}_c^2)}. \tag{15}$$

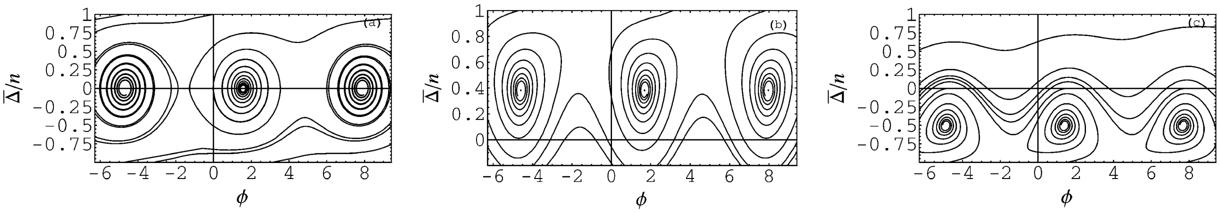

Fig. 2. Numerical simulations for the averaged system at $\omega =$ (a) 3.83, (b) 3.7 and (c) 4.0 where the latter two are for the cases of detuning and the corresponding values for the fixed points, $\bar{\Delta}_c$ are 0.0, 0.383, and -0.502, respectively.

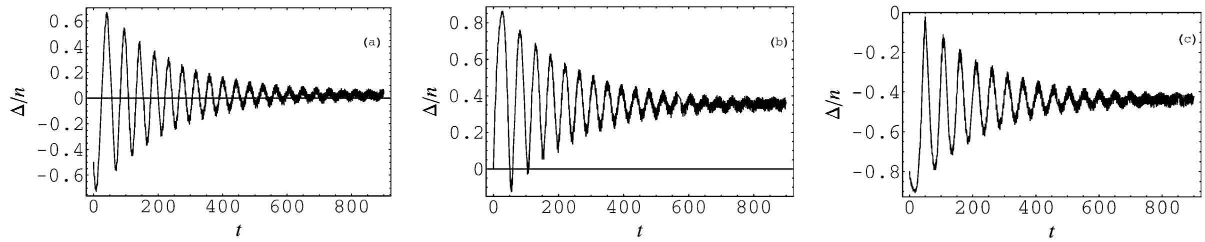

Fig. 3. Numerical simulations for the full system at $\omega =$ (a) 3.83, (b) 3.7 and (c) 4.0. We note the profile of the curves decaying asymptotically to values 0.355, 0.02 and -0.44, respectively.

In the limiting case $\Delta^2 \ll n^2$ we can obtain explicit expressions for the fixed points

$$\bar{\Delta}_c = \frac{\sigma_0(\delta_0 - \omega)}{(C_0 + C_1 - 4C_{01}) + fC_{01}(1 - \eta^2\bar{\Delta}_c^2/(2f^2C_{01}^2n^2)}, \tag{16}$$

$$\cos(\phi_c) \approx -\frac{2\sigma_0\eta\bar{\Delta}_c}{C_{01}fn^2}. \tag{17}$$

In the case of damping term $-\epsilon\theta_t$ we get instead of (15) the expression

$$\cos(\phi_c) = -\frac{2\sigma_0\epsilon\omega}{fC_{01}(n^2 - \bar{\Delta}_c^2)}. \tag{18}$$

Interestingly, the expression for Δ_c (14) is not changed.

Let us consider the case of a zero separation between wells $x_0 = 0$. For this case $C_0 \approx 0.565$, $C_1 \approx 0.335$, $C_{01} \approx 0.16$, and $\beta_0 = -1.55$, $\beta_1 = 1.4$, $\delta_0 = 3.83$. Let us consider $f_0 = 0.5$, $\omega = 3.83$, and $\eta = 0.01$. Then the fixed point is 0.0. For $\omega = 3.7$, $\Delta_c = 0.383$. In Fig. 2 the results of the numerical simulations of the averaged system are shown and in Fig. 3—for the full system (9) the corresponding behaviour are presented.

4. Resonances in the self-trapping regime

Let us study the influence of the parametric drive and damping on the oscillations in the self-trapping regime. Fixed points are defined by Eqs. (14) and (18). We will consider below the case of the damping term of the form $-\epsilon\theta_t$. It is useful to look for the solution of the system (9) by expanding it near the fixed point:

$$\Delta = \Delta_c + \Delta_1, \qquad \theta = \theta_c + \theta_1. \tag{19}$$

Substituting this expansion into the system (9) we find, for $\sigma = \sigma_0 + \sigma_1$, $\sigma_0 = \pm 1$

$$\Delta_{1,t} = \sigma_0 C_{01}\big(n^2 - \Delta_c^2\big)\theta_1 - \sigma_1(t)C_{01}\big(n^2 - \Delta_c^2\big)\theta_1 - \epsilon\theta_{1,t}, \tag{20}$$

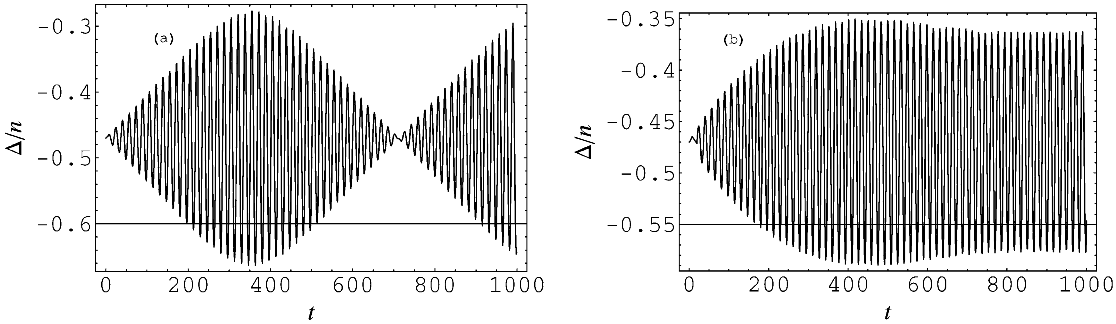

Fig. 4. Numerical simulations for the full system at the main resonance: (a) the damping is zero, (b) the damping is $\epsilon = 0.01$.

$$\theta_{1,t} = \delta_{1,t} - \sigma_1(t)\alpha\Delta_1 - \sigma_0\alpha\Delta_1 - \sigma_1(t)\alpha\Delta_1. \tag{21}$$

Neglecting small second order terms we get the closed equation for Δ_1

$$\Delta_{1,tt} + \omega_\Delta^2\Delta_1 + \epsilon|\alpha|\Delta_{1,t} = -\frac{\omega_\Delta^2}{|\alpha|}\delta_1(t) - \omega_\Delta^2\Delta_c\sigma_1(t) = Q\sin\omega t, \tag{22}$$

where $\alpha = C_0 + C_1 - 6C_{01}$, $\delta_1(t) = \sigma_1(t)((n - 5N_0)C_0 - (n - 5N_1)C_1)$ and $Q = \omega_\Delta^2 f[((n - 5N_0)C_0 - (n - 5N_1)C_1)/|\alpha| + \Delta_c]$. Let us consider the periodic modulation in time $\sigma_1 = f\sin(\omega t)$. The main contribution occurs in the main resonance region $\omega = \omega_\Delta$. The inspection of the full system shows that the parametric resonance at $\omega = 2\omega_\Delta$ has the much weaker instability rate. Looking for the solution of the form $\Delta_1 = A\sin(\phi)$ we obtain that the amplitude at the resonance is

$$A = \frac{Q}{\sqrt{(\omega^2 - \omega_\Delta^2)^2 + \epsilon^2\alpha^2\omega^2}}. \tag{23}$$

For example, for the parameters $C_0 = 0.211$, $C_1 = 0.245$, $C_{01} = 0.22$, $n = 1$, $\epsilon = 0.01$, $\delta_0 = 0.406$, and $f = 0.03$ we get $\Delta_c = -0.47$, $\omega_\Delta = 0.385$ and that $A \approx 0.12$, while the theory gives $A = 0.11$. The oscillations at the main resonance are shown in Fig. 4.

When the amplitude of modulations f grows the trapped phase regime is switched to the running phase regime. The critical value of f can be estimated from the inequality $A \geqslant b\Delta_c$, and $b \geqslant 1$. Then, we obtain the estimate for critical f as

$$f_{\rm crit} \approx \frac{b\Delta_c\sqrt{(\omega^2 - \omega_\Delta^2) + \epsilon^2\alpha^2\omega^2}}{Q_0}, \tag{24}$$

where $Q_0 = Q/f$.

For the set of parameters $C_0 = C_1 = 0.22$, $C_{01} = 0.26$, $n = 1$, $\beta_1 = 0.05$, $\beta_0 = 0$, $N_0 = 5$ the numerical simulation of the full system shows that $f_{\rm crit} \approx 0.5$ for $\epsilon = 0$; $f_{\rm crit} \approx 0.65$, $\epsilon = 0.001$; $f_{\rm crit} \approx 0.75$, $\epsilon = 0.01$, $f_{\rm crit} \approx 1.5$, $\epsilon = 0.1$, which agrees well with this estimate for $b \approx 5$.

When the amplitude of oscillations grows, the linear approximation fails, and the nonlinear effects start to dominate. The results of the numerical simulations for the amplitude–frequency characteristics are given in Fig. 5. It can be seen that for $f \leqslant 0.05$ we have good agreement with the linear approximation and for larger values of f, deviations are introduced. The theory describing the nonlinear resonances in the macroscopic quantum tunnelling requires a separate investigation.

The numerical simulations of nonstationary GP equation with the double-well potential shows, that the coupled-mode theory describes well the running phase regime. For the MQST regime it describes well the position of MQST states, while the amplitude of oscillations are underestimated. According to this remark our results on

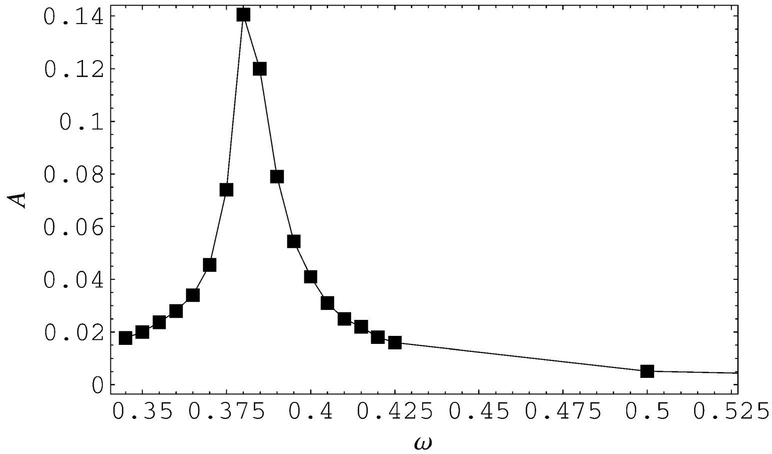

Fig. 5. The amplitude–frequency curve as a function of ω with the parameters $C_0 = 0.211$, $C_{01} = 0.22$, $C_1 = 0.245$, $\epsilon = 0.01$ and $f = 0.03$.

the transition from the MQST regime to the running phase regime should be considered as underestimating the transition threshold.

5. Conclusion

In conclusion, we have investigated the nonlinear tunnelling in BEC in double-well potential, when the atomic scattering length a_s is varied in time and the damping effects are taken into account. The coupled-mode theory is applied for the analysis. We predict the existence of stable running phase regimes in the tunnelling with the dissipation and the periodic variation of a_s. The resonances between oscillating scattering length and the atomic imbalance oscillations between modes are also analyzed. This effect can lead to the resonant enhancing of the tunnelling between modes in the double-well potential, so the control of oscillations of atomic population becomes possible. The switching from the self-trapping regime to the running phase regime under periodic modulations of a_s is predicted.

It is of interest to extend this analysis to the case of high densities. It requires the investigation of models which are beyond the NLS equation limit [16,17] and such analysis will be considered elsewhere.

Acknowledgements

F.Kh.A. is grateful to Research Center of IIUM, Kuala Lumpur for the partial financial support his work and for the hospitality.

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